Wiener-Hopf kernel estimation

NEU 466M
Spring 2020
Problem setup

\[
\{ \cdots x_{t-1}, x_t, x_{t+1} \cdots \} \quad \text{time-varying signal (stimulus)}
\]
\[
\{ \cdots y_{t-1}, y_t, y_{t+1} \cdots \} \quad \text{time-varying signal (response)}
\]

Assume \( y \) derived from \( x \), through convolution with unknown kernel \( h \) and small noise term \( \epsilon \):

\[
y(n) = \sum_{m=M_1}^{M_2} x(n - m) h(m) + \epsilon(n)
\]

\( \{ h_{M_1}, \cdots, h_{M_2} \} \) unknown kernel. If \( M_1 = 0 \): causal.
Question

\[
y(n) = \sum_{m=M_1}^{M_2} x(n - m)h(m) + \epsilon(n)
\]

What is the best possible estimate of \( h \), given \( x, y \)?
Minimization problem

\[ E = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left( y_n - \sum_{m=M_1}^{M_2} x_{n-m} h_m \right)^2 \]

\[ \hat{h}_{M_1} \cdots \hat{h}_{M_2} = \arg \min_{h_{M_1} \cdots h_{M_2}} E \]

Find \( h \) that minimizes the squared error between measured \( y \) and values predicted from \( x \) by the model.
Minimization problem

\[ E = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left( y_n - \sum_{m=M_1}^{M_2} x_{n-m} h_m \right)^2 \]

\[ \hat{h}_{M_1} \cdots \hat{h}_{M_2} = \arg \min_{h_{M_1} \cdots h_{M_2}} E \]

Compare: (very similar) linear regression framework.
Solve: minimization problem

\[ 0 = \frac{\partial E}{\partial h_i} = - \sum_{n} \left( y_n - \sum_{m=M_1}^{M_2} x_{n-m} h_m \right) x_{n-i} \]

\[ = - \sum_{n} y_n x_{n-i} + \sum_{m} (h_m \sum_{n} x_{n-m} x_{n-i}) \]

\[ = C_{xy}^i + \sum_{m=M_1}^{M_2} h_m C_{i-m}^{xx} \]
Wiener-Hopf equations

\[ C_{xy}^i = \sum_{m=M_1}^{M_2} h_m C_{xx}^{i-m} \]

- This is the least-squares optimal solution for the unknown kernel \( h \).
- It depends on the cross-correlation of the input and the response (STA).
- But it also depends on the auto-correlation of the input, unlike the STA.
Linear regression as special case of Wiener-Hopf

\[ C_{xy}^i = \sum_{m=M_1}^{M_2} h_m C_{i-m}^{xx} \]

No time-lags in auto- and cross-correlation since \(x, y\) independent samples not time series (so \(i=0\)). Only one term \(h\) (the slope between \(x, y\)), no convolution, so \(M_1 = M_2 = 0\)

\[ C_{xy} = C_{xx} h_0 \]

\[ h_0 = \frac{C_{xy}}{C_{xx}} \]

Optimal least-squares estimate of slope in linear regression (look back at notes)
Wiener-Hopf equations: solution?

\[ C_{i}^{xy} = \sum_{m=M_1}^{M_2} h_m C_{i-m}^{xx} \]

\((M_2-M_1+1) \times 1\) cross-correlation/STA

unknown kernel: \((M_2-M_1+1) \times 1\)

input auto-correlation

\(M_2-M_1+1\) unknowns \(h_m\).

\(M_2-M_1+1\) equations: \(i^{th}\) equation obtained by differentiating w.r.t. \(h_i\).

Thus, generically, a solution exists.

Easy way to solve?