Excitatory heterogeneous intensity-based model

Network topology with $k$ neurons

Stochastic equation:

$$\lambda_i(t) = \lambda_i(0) + \frac{1}{\epsilon_i} \int_0^t (b_i - \lambda_i(s)) \, ds + \sum_{j \neq i} \mu_{ij} \int_0^t N_j(ds) + \int_0^t (b_i - \lambda_i(s)) N_i(ds)$$

Infiniteimal generator: test function $f: (\mathbb{R}^+)^k \rightarrow \mathbb{R}$

$$\mathcal{L}[f](\lambda_1, \ldots, \lambda_k) = \sum_i \frac{b_i - \lambda_i}{\epsilon_i} \frac{\partial}{\partial \lambda_i} f(\lambda_1, \ldots, \lambda_k)$$

$$+ \sum_i \left[ f(\lambda_i + \mu_i) - f(\lambda_i) - \lambda_i \mu_i \right] \lambda_i$$

If interactions are equivalent to a smooth drift $\alpha_i(t)$ then Fokker-Planck equation reads

$$\partial_t \pi_i(\lambda_i, t) = -\partial_{\lambda_i} \left[ \frac{b_i - \lambda_i}{\epsilon_i} + \alpha_i(t) \right] \pi_i(\lambda_i, t) - \lambda_i \pi_i(\lambda_i, t) + \int_0^\infty \lambda \pi_i(\lambda, t) \, d\lambda S_\lambda(\lambda_i)$$
Thermodynamic mean-field limit

A single neuron in a population of \( N \) neurons with interpopulation connectivity \( \frac{\mu_{ij}}{N} \) is subjected to \( \beta_i(t) \).

\[
\alpha_i(t) = \frac{1}{N} \sum_{j \neq i} \sum_{m=1}^{N} \mu_{ij} \Re \langle \tilde{N}_j \rangle(t) \rightarrow \sum_{m=1}^{N} \mu_{ji} \int_0^{+\infty} \lambda \rho_i(\lambda, t) d\lambda
\]

law of large number

The stationary Fokker-Planck equation is solved by

\[
p_i(\lambda) = \frac{e^{\frac{\lambda^2}{2} - \frac{\lambda_i^2}{2} - \frac{\lambda_i^2}{2}}}{15i - \lambda} \left| \frac{\lambda_i^2}{\lambda_i - \lambda} \right| \beta_i z_i \mathbb{1}_{[\lambda_i, \infty]}(\lambda)
\]

Normalization condition - \( \int p_i(\lambda) d\lambda = 1 \) \( \beta_i = F(\beta_i) \)

input/output relation

Thermodynamic mean-field approximation may fail when the dynamics is correlation dominated or when finite size effects are not negligible.
Replica mean-field limit

Get rid of correlations but keep finite-size effect (adapted to sparse network).

\[ R \text{-replica model are made of } R \text{ identical copies of } K \text{ neurons. When a neuron from replica } r \text{ spikes, it delivers interaction to randomly chosen target neuron across replicas.} \]

The replica-mean-field limit is obtained by taking \( R \rightarrow \infty \).

The probability for two replica to interact in finite time vanishes with \( R \rightarrow \infty \). Neurons become asymptotically independent.

Self-consistency equations:

\[ \beta_i \quad \mu_{ij} \quad \beta_j \]

\[ \beta_k \rightarrow \mu_{jk} \rightarrow \text{neuron } i \rightarrow \beta_i \]

independent inputs  \quad \text{outputs (non Poisson)}
Generating Function: Formalism

Characterizing stationary state via functional transform, typically the moment-generating function, \( u \rightarrow \mathbb{E}[e^{uX}] = L(u) \).

Equations specifying \( L_i(\cdot) \) are most conveniently obtained via rate conservation principles.

We have:

\[
e^{u_i(t)} = e^{u_i(0)} + \frac{\mu_i}{\tau_i} \int_0^t (b_i - \lambda_i(s)) e^{u_i(s)} \, ds
\]

\[
+ \sum_{i \neq i} \left( e^{u_i(0)} - 1 \right) \int_0^t e^{u_i(s)} N_i(ds)
\]

\[
+ \int_0^t (e^{u_i(s)} - e^{u_i(0)}) N_i(ds)
\]

Taking expectation with respect to the stationary measure for which \( \mathbb{E}[e^{u_i(t)}] = \mathbb{E}[e^{u_i(0)}] \), leads to:

\[
\frac{\mu_i}{\tau_i} \mathbb{E}\left[(b_i - \lambda_i) e^{u_i(t)}\right] = \sum_{i \neq i} \left( e^{u_i(0)} - 1 \right) \mathbb{E}\left[ \int_0^t e^{u_i(s)} N_i(ds) \right]
\]

\[
+ \int_0^t \mathbb{E}\left[(e^{u_i(s)} - e^{u_i(0)}) N_i(ds) \right]
\]

Palm calculus:

\[
= \sum_{i \neq i} \left( e^{u_i(0)} - 1 \right) \beta_i \mathbb{E}_0 \left[ e^{u_i(0)} \right]
\]

\[
+ \beta_i \mathbb{E}_0 \left[ e^{u_i(0)} - e^{u_i(0)} \right]
\]
The parameters $\beta_i$ have been introduced as free parameters. Solving the replica-mean-field problem amounts to specifying the parameters $\beta_i$. 

$$\frac{\partial}{\partial t} E[e^{\mu_i}] - E[\chi_i e^{\mu_i}] = \sum_{j \neq i} (e^{\mu_j} - 1) \beta_j \frac{\partial}{\partial t} E[e^{\mu_i}]$$

$$+ \beta_i (e^{\mu_i} - E[e^{\mu_i}])$$

Papangelou Theorem

$$= \sum_{j \neq i} (e^{\mu_j} - 1) E[\chi_j e^{\mu_i}]$$

$$+ \beta_i e^{\mu_i} - E[\chi_i e^{\mu_i}]$$

Replica-mean-field

$$= \sum_{j \neq i} (e^{\mu_j} - 1) \beta_j E[e^{\mu_i}]$$

$$+ \beta_i e^{\mu_i} - E[\chi_i e^{\mu_i}]$$

$$L_i(u) = E[e^{\mu_i}], \quad \frac{\partial}{\partial u} L_i(u) = E[\chi_i e^{\mu_i}]$$

System of ODE: 

$$-(1 + \frac{\mu}{\tau_c}) \frac{\partial}{\partial u} L_i + (\frac{u \beta_i}{\tau_i} + \sum_{j \neq i} (e^{\mu_j} - 1) \beta_j) L_i + \beta_i e^{\mu_i} = 0$$

$$u = 0 \Rightarrow -\frac{\partial}{\partial u} L_i(0) + \beta_i = 0 \quad \checkmark$$

$$L_i(u) = 1$$

The parameters $\beta_i$ have been introduced as free parameters. Solving the replica-mean-field problem amounts to specifying the parameters $\beta_i$. 

$$L_i(u) = 1$$
In principle, the replica-mean-field ODE admits an infinity of solutions. However, "physical solutions", which corresponds to a probabilistic model, satisfies strong regularity properties. In particular, they analytical function.

ODE of the form: 

\[-(1 + \frac{u}{z_i}) \partial_u L_i + F_i(u) L_i + g_i(u) = 0\]

\[F(-z_i) > 0\]

Simple analytical considerations show that there is a unique continuous (analytic) solution:

\[L_i(u) = \int_{-z_i}^{u} e^{-\frac{\int_{v}^{u} F_i(w) \, dw}{1 + \frac{u}{z_i}}} \frac{g_i(v) \, dv}{1 + \frac{v}{z_i}}\]

\[\uparrow\]

choice of the bound dictated by continuity

\[\beta_i\] obtained from normalization condition: \[L_i(0) = 1\]